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UDC 539. 3

CYLINDRICAL BENDING OF A PLATE BY RIGID STAMPS

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A periodic contact problem of the cylindrical bending of a plate by rigid stamps is considered from the aspect of equations of elasticity theory as well as Kirchhoff-Love theory with and without transverse compression of the material in the contact zone taken into account. Analysis of the solutions obtained permits illumination of the question of the error and of the possibility of using the classical theory of plates and shells in analyzing contact problems. A comparative analysis is given of the nature of the distribution and of the magnitude of the stresses of the plate in the contact zone, of the character of the contact reaction distribution and the dependence between the magnitude of the contact zone and the force applied to the stamp. The apparatus of integral equations is used in considering the problem from the aspect of elasticity theory, while the solution is obtained in closed form by means of Kirchhoff theory.

An analogous problem on the basis of the elasticity theory equations has been solved in [1] also by a method different from that elucidated below. However, only sufficiently thick plates (the ratio between the thickness and the characteristic dimension is not less than $^1/_{20}$) are considered there. But a comparison between the stresses obtained when using different theories can yield the most correct answer about the applicability of any theory.

1. Solution on the basis of elasticity theory equations. Let us consider an infinite plate of thickness h (Fig. 1) occupying the xz plane and loaded by a system of rigid stamps. The stamps are identical and arranged with a constant spacing $2l$, have a cylindrical base surface so that the contact occurs over the whole length along

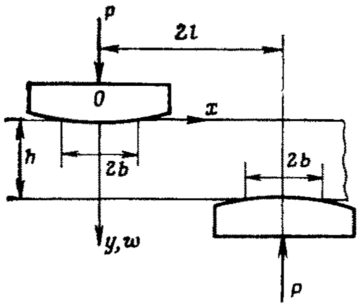

Fig. 1

the z-axis and the plate is under plane strain conditions.

Let us first write the solution for a self-balanced system of normal concentrated forces P_1 applied at the points $x = 0, \pm 2l, \pm 4l, \ldots$. Using the general solution [2] and the boundary conditions for the stresses

$$\sigma_y = -\frac{P_1}{2l}\left(\frac{1}{2} + \sum_{n=1}^{\infty} \cos n\varphi\right) \text{ for } y = 0 \tag{1.1}$$

$$\sigma_y = -\frac{P_1}{2l}\left(\frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n \cos n\varphi\right) \text{ for } y = h$$

$$\tau_{xy} = 0 \quad \text{for } y = 0, h; \quad \varphi = \pi x / 2l$$

we obtain

$$\sigma_x = \frac{P_1}{2l}\sum_{m=1}^{\infty} [\psi''_{1m}(t) + (-1)^m \psi''_{3m}(t)] \cos m\varphi \tag{1.2}$$

$$\sigma_y = -\frac{P_1}{2l}\left\{\frac{1}{2} + \sum_{m=1}^{\infty} [\psi_{1m}(t) + (-1)^m \psi_{3m}(t)] \cos m\varphi\right\}$$

Here

$$t = \frac{m\pi y}{2l}, \quad \psi''_{jm} = \frac{d^2\psi_{jm}}{dt^2}$$

$$\psi_{jm}(t) = a_{jm}\,\mathrm{ch}\, t + b_{jm}\,\mathrm{sh}\, t + c_{jm} t\,\mathrm{ch}\, t + d_{jm} t\,\mathrm{sh}\, t$$

$$a_{1m} = 1, \quad a_{3m} = 0, \quad c_{1m} = (\mathrm{sh}\,\gamma\,\mathrm{ch}\,\gamma + \gamma)\, a_m^{-1}$$

$$c_{3m} = -(\gamma\,\mathrm{ch}\,\gamma + \mathrm{sh}\,\gamma)\, a_m^{-1}, \quad d_{1m} = -\mathrm{sh}^2\gamma\, a_m^{-1}, \quad d_{3m} = \gamma\,\mathrm{sh}\,\gamma\, a_m^{-1}$$

$$b_{1m} = -c_{1m}, \quad b_{3m} = -c_{3m}, \quad a_m = \mathrm{sh}^2\gamma - \gamma^2, \quad \gamma = m\pi h / 2l$$

We find the deflection w of the plate surface $y = 0$ in the y direction by integration of the Hooke's law relationship for plane strain

$$\frac{\partial w}{\partial y} = \frac{1}{E_1}(\sigma_y - \nu_1\sigma_x), \quad E_1 = \frac{E}{1-\nu^2}, \quad \nu_1 = \frac{\nu}{1-\nu}$$

Taking account of (1.2), we obtain after extracting the divergent part of the series (a_1 is an arbitrary constant of integration)

$$w = -\frac{2P_1}{\pi E_1}\left(\ln\left|2\sin\frac{\varphi}{2}\right| - \sum_{m=1}^{\infty} b_m \cos m\varphi + a_1\right) \tag{1.3}$$

$$b_m = \frac{1}{m}[c_{1m} - 1 + (-1)^m c_{3m}]$$

The expression (1.3) is used as a Green's function for the deflection of the plate contact zone due to reaction. On the other hand, if the curvature of the stamp base is $\varkappa = 1 / R = \mathrm{const}$, then the deflection of the plate contact zone in the case of a close fit will be (d is the unknown deflection of the stamp)

$$w = d - 2l^2\varphi^2 / (\pi^2 R)$$

Equalizing the designated deflections results in an integral equation to determine the normal reaction q of the stamp

$$\int_{-\theta}^{\theta} q_1 \ln\left|2\sin\frac{\varphi-\varphi_0}{2}\right| d\varphi_0 = \int_{-\theta}^{\theta} q_1 K(\varphi-\varphi_1)\,d\varphi_1 + \varphi^2 + \delta \qquad (1.4)$$

$$(-\theta < \varphi < \theta)$$

$$q = \frac{E_1 l}{2R} q_1, \quad K(\varphi-\varphi_1) = \sum_{m=1}^{\infty} b_m \cos m(\varphi-\varphi_1)$$

Here 2θ is the magnitude of the contact zone and δ is an arbitrary constant characterizing the deflection of the stamp.

Let us convert (1.4) to a Fredholm equation of the second kind by inverting the integral in the left side [3].

Denoting the right side of (1.4) by $f(\varphi)$ for brevity, we obtain

$$q_1(\varphi_0) = \frac{1}{2\pi^2 X(\varphi_0)} \int_{-\theta}^{\theta} \frac{X(\varphi) f'(\varphi)\,d\varphi}{\sin{}^1/_2(\varphi-\varphi_0)} + \frac{A}{\pi}\frac{\cos{}^1/_2\varphi_0}{X(\varphi_0)}, \qquad (1.5)$$

$$X(\varphi) = \sqrt{2(\cos\varphi - \cos\theta)}$$

Here A is a constant determined from the condition

$$A \ln\sin\frac{\theta}{2} = \frac{1}{\pi}\int_{-\theta}^{\theta} \frac{f(\varphi)}{X(\varphi)}\cos\frac{\varphi}{2}\,d\varphi \qquad (1.6)$$

Since the function $f(\varphi)$ depends on the arbitrary constant δ, then according to (1.6) the constant A can be considered arbitrary and condition (1.6) itself is not taken into account, since it is satisfied for any A by the selection of δ.

Let us select the constant A so that the reaction q_1 would be bounded at the ends of the contact zone. To do this, let us find the condition for the right side of (1.5) to vanish for $\varphi = \pm\theta$. Using the identities

$$X(\varphi) = \frac{X^2(\varphi_0)}{X(\varphi)} + \frac{2(\cos\varphi_0 - \cos\varphi)}{X(\varphi)},$$

$$\cos\varphi_0 - \cos\varphi = 2\sin\frac{\varphi_0-\varphi}{2}\sin\frac{\varphi_0+\varphi}{2}$$

we rewrite (1.5) as

$$q_1(\varphi_0) = \frac{X(\varphi_0)}{2\pi^2}\int_{-\theta}^{\theta}\frac{f'(\varphi)\,d\varphi}{X(\varphi)\sin{}^1/_2(\varphi-\varphi_0)} + \frac{A}{\pi}\frac{\cos{}^1/_2\varphi_0}{X(\varphi_0)} -$$

$$\frac{1}{2\pi^2 X(\varphi_0)}\int_{-\theta}^{\theta}\frac{4\sin{}^1/_2(\varphi+\varphi_0) f'(\psi)\,d\varphi}{X(\varphi)}$$

The requirement that the sum of the last two members vanish yields the two conditions

$$\int_{-\theta}^{\theta}\frac{f'(\varphi)\cos{}^1/_2\varphi\,d\varphi}{X(\varphi)} = 0, \quad A = \frac{2}{\pi}\int_{-\theta}^{\theta}\frac{f'(\varphi)\sin{}^1/_2\varphi\,d\varphi}{X(\varphi)}$$

The second condition can be considered satisfied; it is the formula to determine A while the first is satisfied automatically because of the evenness of the function $f(\varphi)$.

The evenness follows from an elementary analysis of (1. 4).

Thus

$$q_1(\varphi_0) = \frac{X(\varphi_0)}{2\pi^2} \int_{-\theta}^{\theta} \frac{f'(\varphi)\,d\varphi}{X(\varphi)\sin {}^1/_2(\varphi-\varphi_0)} \tag{1. 7}$$

It can be established from (1. 4) that the function q_1 is even, hence, its even part $K(\varphi, \varphi_1)$ can henceforth be taken in place of the kernel $K(\varphi - \varphi_1)$. Inserting the function $f(\varphi)$ in (1. 7) after this (let us recall that $f(\varphi)$ is the right side of (1. 4)), evaluating the appropriate integrals and introducing the new unknown function $y(\varphi)$ by means of the relationship

$$q_1(\varphi) = X(\varphi)\cos\frac{\varphi}{2}\,y(\varphi) \tag{1.8}$$

we obtain the following Fredholm integral equation of the second kind for $y(\varphi)$:

$$y(\varphi_0)\cos\frac{\varphi_0}{2} - \frac{1}{\pi}\int_{-\theta}^{\theta} R(\varphi_1, \varphi_0)\,X(\varphi_1)\cos\frac{\varphi_1}{2}\,y(\varphi_1)\,d\varphi_1 = \frac{1}{\pi}\,b(\varphi_0) \tag{1. 9}$$

Here ($P_n = P_n(\cos\theta)$ are Legendre polynomials, b_k are coefficients defined in (1. 3))

$$R(\varphi_1, \varphi_0) = \frac{1}{2\pi}\int_{-\theta}^{\theta} \frac{dK(\varphi, \varphi_1)}{d\varphi}\,\frac{d\varphi}{X(\varphi)\sin {}^1/_2(\varphi-\varphi_0)} = \tag{1. 10}$$

$$-\sum_{k=1}^{\infty} k b_k \omega_k(\varphi_0)\cos k\varphi_1$$

$$b(\varphi_0) = \frac{1}{\pi}\int_{-\theta}^{\theta} \frac{\varphi\,d\varphi}{X(\varphi)\sin {}^1/_2(\varphi-\varphi_0)}, \tag{1. 11}$$

$$\omega_k(\varphi_0) = \sum_{n=0}^{k-1} P_n \cos\left(n - k + \frac{1}{2}\right)\varphi_0, \quad k = 1, 2, \ldots$$

The numerical solution of (1. 9) is given below. By knowing it we can determine the normal stresses σ_x in a plate, which are of greatest interest in estimating the errors induced in the solution of the problem by the Kirchhoff-Love hypothes. . Using the solution (1. 2) for the stress σ_x due to concentrated forces, we obtain the stress due to the reaction q taking account of the second formula in (1. 4), as

$$\sigma_x = \frac{E_1 l}{2\pi R}\sum_{m=1}^{\infty} f_m(t)\int_{-\theta}^{\theta} q_1\cos m(\varphi-\varphi_1)\,d\varphi_1, \quad f_m(t) = \psi_{1m}''(t) + (-1)^m\varphi_{3m}''(t) \tag{1. 12}$$

The stamp equilibrium condition yields an equation connecting the magnitude of the contact zone θ and the force P applied to the stamp

$$P = \frac{E_1 l^2}{\pi R}\int_{-\theta}^{\theta} q_1\,d\varphi \tag{1. 13}$$

If the Kirchhoff-Love hypotheses are considered valid for the plate examined above (Fig. 1) which is loaded by a system of stamps, then it can be identified with a plate occupying the domain $-l \leqslant x \leqslant l$ and simply supported along the edges $x = \pm l$. When a cylindrical stamp with base curvature $\varkappa = 1/R = \text{const}$ acts on the plate, the maximum stresses in the contact zone will be constant over the length of the zone and equal to

$$\sigma_1 = E_1 h \,/\, (2R) \tag{1.14}$$

It is convenient to compare the stresses (1. 12), evaluated on the basis of elasticity theory, with the stresses (1. 14) and to analyze the dimensionless stress parameter

$$\sigma = \sigma_x \,/\, \sigma_1 \tag{1.15}$$

whose deflection from one in the contact zone will characterize the error induced in the solution of the Kirchhoff-Love theory.

Within the framework of the Kirchhoff-Love theory, the dimensionless parameter P^* of the force P applied to the stamp is related to the parameter β of the magnitude of the contact zone ($2b$ is the actual width of the contact zone) by the condition

$$P^* = 1 \,/\, (1 - \beta) \tag{1.16}$$

Here

$$P^* = \frac{PlR}{2D}, \quad D = \frac{Eh^3}{12(1-\nu^2)}, \quad \beta = \frac{b}{l} = \frac{2\theta}{\pi}$$

Comparing the parameters P^* obtained by using (1. 13) and (1. 16), we can obtain an integral characteristic of the error induced by the Kirchhoff-Love theory.

Let us briefly examine the algorithm for numerical solution of the problem. Equation (1. 9) has been solved by reducing it to a system of elgebraic equations by using the quadrature formula

$$\frac{1}{\pi}\int_{-\theta}^{\theta} \frac{f(\alpha)\, d\alpha}{X(\alpha)} = \frac{1}{n}\sum_{k=1}^{n} \frac{f(\alpha_k)}{\cos {}^1/_2\alpha_k}, \quad \alpha_k = -2 \arcsin\left(\sin\frac{\theta}{2}\cos\frac{2k-1}{2n}\pi\right) \tag{1.17}$$

which is obtained from the Meller quadarture formula [4] by a simple change of variable and is exact for a polynomial of degree up to $2n-1$. Formula (1. 17) has also been used to evaluate the integrals (1. 12), (1. 13) and (1. 11). The last integral is first regularized by using the identity

$$\int_{-\theta}^{\theta} \frac{d\varphi}{X(\varphi)\sin {}^1/_2(\varphi - \varphi_0)} = 0$$

2. Solution by means of Kirchhoff theory taking account of transverse compression of the plate. The numerical computations presented here permit making a deduction about the possibility of using Kirchhoff theory for sufficiently thin plates. However, the Kirchhoff theory contains some formal contradictions. Thus, concentrated forces at the ends of the contact zone appear in the composition of the reaction as a result of a jump in the transverse force. The dependence between the force applied to the stamp and the magnitude of the contact zone is isolated in the initial contact stage, since the contact zone does not appear at once but only as the force grows to some value.

The formal contradictions can be eliminated within the framework of approximate theories either by taking account of the transverse shear strain [5] or by transverse compression of the plate, or of the two together. A solution taking account of transverse compression is presented below, but without taking account of transverse shears. When taking account of transverse shear, this solution eliminates the concentrated forces at the ends of the contact zone and the discrepancy noted between the applied force and the magnitude of the contact zone in the initial stage. Moreover, the character of the change in the reaction along the length of the contact zone for medium and small zones is close to the true value while taking account of just the transverse shear strain yields a reaction which has a maximum value at the ends of the contact zone, where it should really be zero.

We find the magnitude of the transverse compression by applying a known, but formally invalid within the framework of the Kirchhoff theory, method of integrating the Hooke's law relationship for the transverse strain ε_y after having found the stress σ_y from the equilibrium equation. If the difference between the deflection $w(0)$ of the contact surface and the deflection of the plate middle surface is understood to be transverse compression, we find

$$w(0) = w + \frac{13}{32}\frac{h}{E}q + \frac{\nu h^2}{8(1-\nu)}\frac{d^2w}{dx^2} \tag{2.1}$$

where q is the reaction from the stamp on the plate in the y direction (Fig. 1). The last member in (2. 1) is discarded for simplicity, since it does not spoil the qualitative picture of the solution and, as computations taking this member into account have shown, does not alter the solution quantitatively, in practice.

If the curvature of the stamp base is $\varkappa = 1 / R = \text{const}$ as above, then the condition for the stamp and plate to fit closely in the contact zone will be (d is the stamp deflection)

$$w + \frac{13}{32}\frac{h}{E}q = d - \frac{\varkappa x^2}{2} \tag{2.2}$$

Condition (2. 2) can be considered as a formula to determine the deflection w if the reaction is known. We obtain the equation for the reaction if we substitute the deflection from (2. 2) into the plate equilibrium equation in the contact zone

$$d^4w / dx^4 = q / D$$

We consequently obtain the equation

$$\frac{d^4q}{d\xi^4} + 4\omega^4 q = 0, \quad \xi = \frac{x}{l}, \quad \omega = \frac{l}{h}\sqrt[4]{\frac{96}{13}(1-\nu^2)} \tag{2.3}$$

In contrast to the solution for the reaction q obtained when taking into account just the transverse shear, and determined from a second order equation, the solution of (2. 3) can be subjected not only to stamp equilibrium conditions but also the condition of vanishing reaction at the ends of the contact zone. In the case under consideration of a symmetrical stamp, this solution ($\beta = b / l$ is the dimensionless value of the contact zone)

$$q = \frac{\omega P}{l}\frac{f(\xi)}{\operatorname{sh}\omega\beta - \sin\omega\beta} \tag{2.4}$$

$$f(\xi) = \operatorname{sh}\omega(\beta+\xi)\sin\omega(\beta-\xi) + \operatorname{sh}\omega(\beta-\xi)\sin\omega(\beta+\xi)$$

It follows from (2. 4) that the form of the solution is independent of the plate support conditions; these conditions will only influence the dependence between the force P and the magnitude of the contact zone β.

Let us note that there are two arbitrary quantities in the expression for the deflection in the contact zone, which is determined from (2. 2) in terms of the reaction (2. 4), namely, the stamp deflection d and the magnitude of the contact zone β. The general solution for the deflection outside the contact zone will contain four arbitrary quantities. These six arbitrary quantities permit compliance with four juncture conditions at the end of the contact zone and two boundary conditions at the end of the plate $x = l$.

We therefore obtain a relationship between the force P pressing the stamp and the magnitude of the contact zone β as well as between the deflection d and β. In the case of simply supported plate edges, we have (P^* is the external loading parameter exactly as in (1. 16))

$$P^* = \frac{1}{(1-\beta)\psi}, \quad a^* = 1 - \frac{1}{3}(1-\beta)^2\frac{\psi_1}{\psi}$$

$$a^* = \frac{2dR}{l^2}, \quad \psi = 1 + \frac{1}{\omega(1-\beta)}\frac{\operatorname{ch} 2\omega\beta + \cos 2\omega\beta}{\operatorname{sh} 2\omega\beta - \sin 2\omega\beta}$$

$$\psi_1 = 1 + \frac{3}{\omega(1-\beta)}\frac{\operatorname{ch} 2\omega\beta + \cos 2\omega\beta}{\operatorname{sh} 2\omega\beta - \sin 2\omega\beta} + \frac{3}{\omega^2(1-\beta)^2}\frac{\operatorname{sh} 2\omega\beta + \sin 2\omega\beta}{\operatorname{sh} 2\omega\beta - \sin 2\omega\beta}.$$

3. Let us estimate the accuracy of the numerical solution in Sect. 1 and the results. The accuracy depends on the number of terms kept in the series (1. 10), determining the kernel of (1. 9), and the number n in the quadrature sum of type (1. 17). It has been established by variation of these quantities that the error in the reaction does not exceed two percent if the number of terms in the series (1. 10) is taken equal to the magnitude of the parameter $2l / h$. The error in the stresses will hence be still smaller. As regards the number n in the sum (1. 17), it is sufficient to take $n = 20$. The results for $n = 20$ and 40 agree even in the case of a thin plate ($2l / h = 100$). For $2l / h < 100$ the reaction will be a smoother function and the accuracy will grow.

$2\,l/h$	20		100		
β	y	O	y	O	k
0.03	0.819	0.720	1.017	1.018	1.031
0.06	0,970	0.981	1.049	1.050	1.064
0.1	1.032	1.045	1.095	1.096	1.111
0.3	1.303	1.313	1.402	1.404	1.429
0.6	2.140	2.166	2.417	2.425	2.500
0.8	3.740	3.821	4.701	4.709	5.000

The change in the load parameter P^* is shown in the table as a function of the magnitude of the contact zone $\beta = b / l$. The columns y hence correspond to the solution in Sect. 1, the columns O to the solution in Sect. 2, and k to the solution using Kirchhoff-Love theory without taking account of transverse compression.

It is seen that a discrepancy between the Kirchhoff and elasticity theories occurs only for very small and very large contact zones. Taking account of transverse compression improves the result.

Shown in Fig. 2 is the character of the measurement of the dimensionless reaction $q^* = 2ql\beta / P$ along the length of the contact zone ($\xi = x / b$, Fig. 1) for the following values of $2\,l / h$ and β : *1* — $2l / h = 20$, $\beta = 0{,}1$, *2* — $2l / h = 20$, $\beta = 0.3$, *3* — $2l / h = 100$, $\beta = 0.1$, *4* — $2l / h = 100$, $\beta = 0.3$. The solid lines correspond to the solution in Sect. 1, and the dashes to the solution by the formulas in Sect. 2. As we see, concentration of the reaction at the end of the contact zone occurs as the plate thickness diminishes. The approximate solution taking into account transverse compression agrees sufficiently well with that obtained by elasticity theory. Shown for comparison in Fig. 2 is the solution from the theory of plates with transverse shear but not transverse compression taken into account. This is the solid monotonically increasing curve which corresponds to $2l / h = 20$, $\beta = 0.3$.

A numerical computation of the stress parameter (1. 15) shows that it differs most strongly from one at the ends of the contact zone. Graphs of its change at the ends of the contact zone are given in Fig. 3 as a function of the magnitude of the zone and the

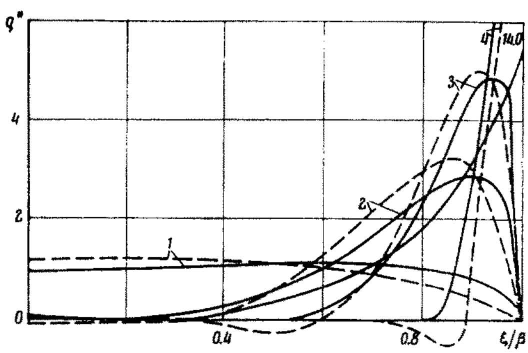

Fig. 2

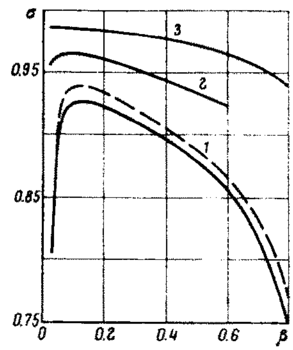

Fig. 3

thickness of the plate. The values of $2l / h = 20, 60, 100$ correspond to curves *1—3* . Shown by dashes is the solution by Kirchhoff theory, but taking account of the nature of the change in reaction in the contact zone, obtained from (2. 4). As we see, the true stresses in the contact zone differ insignificantly for thin plates from those obtained by Kirchhoff theory.

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WAVE EXCITATION IN A LAYER BY A VIBRATING STAMP

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The problem of the vibrations of a rigid circular stamp on the surface of an elastic layer at rest on a rigid base is examined. There is no friction between the stamp and the layer, and between the layer and the base. The contact stresses